Lecture # 11

<u>Recall:</u> X is a discrete random variable if its range is countable.

If X is a DRV, then a <u>probability</u> <u>mass</u> <u>function</u> (pmf) for X is a function $f(x)$ such that:

1) $f(x) \geq \forall x.$

2) $\sum_{x \in range(X)} f(x) = 1.$

3) $f(x) = P(X = x).$

<u>Expected Value of a Random Variable</u>

Let X be a discrete random variable with range $\{x_1, x_2, x_3, \dots\}$. The <u>expected value</u> of X is the average of the possible values of X weighted by their individual probabilities. That is, if X has probability mass function $p(x) = P(X = x)$ then the expected value is

$$\mu = E[X] = \sum_{i=1}^{\infty} x_i\, p(x_i).$$

Note that $E[X]$ need not be in the range of X. For example, suppose our experiment is "flip a coin". So $S = \{h, t\}$. Consider the random variable

$$X : S \to \{0, 1\}.$$
$$: h \mapsto 1$$
$$: t \mapsto 0$$

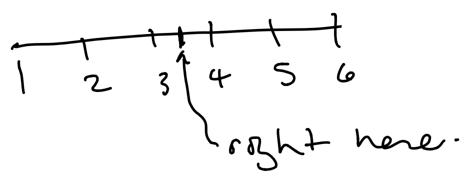
We have a probability distribution

$$P(X=0) = 1/2 \quad P(X=1) = 1/2.$$

and so $E[X] = 0\left(\frac{1}{2}\right) + 1 \cdot \left(1/2\right) = 1/2.$

What is the expected value of a roll of a die?

$$E[X] = \frac{1}{6}(1+2+3+4+5+6) = 3.5.$$



__Ex.__ Consider the following game:

- a fair die is rolled
- if the roll n is 2, 3, 5, you win $2 \cdot n$ dollars
- if the roll $n = 1, 4, 6$, you get nothing.

How much would you pay to play one round of this game?

Answer: We consider the expected value of the game! Let W be the discrete random variable "winnings", so range$(W) = \{0, 2, 4, 6, 10\}$

$$P(W=0) = 3/6 \quad (\text{for a roll } 1, 4, 6)$$

$$P(W=4) = 1/6 \quad (\text{roll } 2)$$

$$P(W=6) = 1/6 \quad (\text{roll } 3)$$

$$P(W=10) = 1/6 \quad (\text{roll } 5).$$

$$\text{So } \mu = E(W) \underset{= \sum f(x)x}{=} \frac{3}{6}0 + 1/6 \cdot 4 + 1/6 \cdot 6 + 1/6 \cdot 10$$

$$= \frac{4+6+10}{6} = \frac{20}{6}$$

$$= \frac{10}{3} \approx \$3.33.$$

So on average, you should expect to win \$3.33, so if you pay more than that per round, you should expect to lose money.

(A game where the cost equals the expected value) is called a fair game.

It also makes sense to have negative expected value, if some of the values of X are negative.

Expectation of a function of a random variable.

Suppose that X is a discrete random variable with range $\{x_1, x_2, x_3 \dots\} \subseteq \mathbb{R}$. If $g: \mathbb{R} \to \mathbb{R}$ is a function. Then $g(X)$ is a discrete random variable as well.

The possible values of $g(X)$ are

$$\text{range}(g(X)) = \{\underbrace{g(x_1), g(x_2), g(x_3) \dots}_{\text{some of these may be repeated if } g \text{ is not 1-1.}}\}.$$

If X has probability mass function $p(x)$, then $g(X)$ has a probability mass function determined by

$$P(g(X) = g(x_i)) = \sum_{x_j : g(x_j) = g(x_i)} p(x_j)$$

Using this, one can show that

<u>Prop:</u> If X is a DRV with range $\{x_1, x_2, \dots\}$. and g is a function with $\text{range}(X) \subseteq \text{dom}(g)$, then
$$E[g(X)] = \sum_{i=1}^{n} g(x_i)\, p(x).$$

In general, $E[g(X)] \neq g(E[X])$. However, for real # a, b, we have

$$\begin{aligned} E[aX + b] &= \sum (a x_i + b)\, p(x_i) \\ &= a \sum x_i p(x_i) + b \underbrace{\sum p(x_i)}_{=1} \\ &= a E[X] + b. \end{aligned}$$

<u>Rmk/Terminology:</u> $E[X]$ is also called the "first moment" of X. $E[X^n] = \sum x^n p(x)$ is called the nth moment of X.

<u>Variance</u>: Given a random var. X, we can summarize much of the properties of X by two numbers. One of them is the mean i.e. the average of all possible values of X. The other is the <u>variance</u>. The variance is a

measure of how spread out the values are.

<u>Ex:</u> $X = \begin{cases} -1 & \text{with prob } 1/2 \\ +1 & \text{with prob } 1/2. \end{cases}$

and $Y = \begin{cases} -100 & \text{with prob } 1/2. \\ +100 & \text{with prob. } 1/2. \end{cases}$

are two D.R.V. with mean 0, but they will have different variances.

<u>Defn:</u> Let X be a DRV. Then

$$\sigma^2 = \mathrm{Var}(X) = E[(X-\mu)^2],$$

where $\mu = E[X]$.

Note that

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X-\mu)^2] \\
&= \sum (x-\mu)^2 p(x) \\
&= \sum (x^2 - 2\mu x + \mu^2)\, p(x). \\
&= \underbrace{\sum x^2 p(x)}_{=E[X^2]} - 2\mu \underbrace{\sum x p(x)}_{=\mu.} + \mu^2 \underbrace{\sum p(x)}_{=1} \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2 = E[X^2] - E[X]^2.
\end{aligned}$$